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FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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1. Introduction

In 1990, Takahashi proved the following nonconvex minimization theorem, which was used to obtain Caristi's fixed point theorem [1], Ekeland's ε-variational principle [3] and Nadler's fixed point theorem [6].

Theorem 1 (Takahashi [8]**).** *Let X be a complete metric space with metric d and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x \in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u, v) \leq f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$.*

This theorem was improved by several authors; see [5], [9] and [10]. On the other hand, Ćirić [2] proved an interesting fixed point theorem for a quasi-contraction which generalizes some fixed point theorems in a complete metric space. Recently Kada, Suzuki and Takahashi introduced the following concept.

Definition ([4]). Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The metric d is a w-distance on X. Other examples of w-distance are stated in [4] and [7]. Using it, Kada, Suzuki and Takahashi [4] generalized Caristi's fixed point theorem, Ekeland's ε-variational principle, Takahashi's nonconvex minimization theorem and Ćirić's fixed point theorem. One of them is the following fixed point theorem.

Theorem 2 ([4]). *Let X be a complete metric space, let p be a w-distance on X and let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that*

$$p(Tx, T^2x) \leq rp(x, Tx)$$

for every $x \in X$ *and*

$$\inf\{p(x,y)+p(x,Tx) : x \in X\} > 0$$

for every $y \in X$ *with* $y \neq Ty$. *Then there exists* $x_0 \in X$ *such that* $x_0 = Tx_0$. *Moreover, if* $z = Tz$, *then* $p(z,z) = 0$.

In this paper, we first give some Examples and Lemmas connected with w-distance. Next we give another proof of a generalization of Theorem 1. Further we prove two fixed point theorems which generalize Ćirić's fixed point theorem. Finally, using them, we give another proof of a characterization of metric completeness.

2. Preliminaries

In this Section, we state, without the proofs, Examples and Lemmas connected with w-distance.

Example 1. Let $X = \mathbb{R}$ be a metric space with the usual metric and let $f, g : X \to [0,\infty)$ be continuous functions such that

$$\inf_{x\in X} \int_x^{x+r} f(u)du > 0 \quad \text{and} \quad \inf_{x\in X} \int_x^{x+r} g(u)du > 0$$

for any $r > 0$. Then a function $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = \begin{cases} \displaystyle\int_x^y f(u)du, & \text{if } x \leq y, \\ \displaystyle\int_y^x g(u)du, & \text{if } y \leq x \end{cases}$$

for every $x, y \in X$ is a w-distance on X.

Example 2 ([4]). Let X be a metric space and let T be a continuous mapping from X into itself. Then a function $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = \max\{d(Tx,y), d(Tx,Ty)\} \quad \text{for every} \quad x, y \in X$$

is a w-distance on X.

Example 3. Let X be a metric space with metric d, let T be a mapping from X into itself such that, for every $x \in X$, the orbit $\{x, Tx, T^2x, \cdots\}$ is bounded. Then a function $p : X \times X \to [0,\infty)$ given by

$$p(x,y) = \sup\{d(T^k x, y) : k \in \mathbb{N} \cup \{0\}\} \quad \text{for every} \quad x, y \in X$$

is a w-distance on X.

Example 4. Let X be a metric space with metric d and let $\{x_n\}$ be a sequence in X such that

(i) $\{x_n\}$ is Cauchy;
(ii) $\{x_n\}$ does not converge;

(iii) $x_i \neq x_j$ if $i \neq j$.

Then a function $p : X \times X \to [0, \infty)$ defined by

$$p(x,y) = \begin{cases} 2^{-i} + 2^{-j}, & \text{if } \ x = x_i \text{ and } y = x_j, \\ 2^{-i} + 1, & \text{if } \ x = x_i \text{ and } y \notin \{x_n\}, \\ 1 + 2^{-j}, & \text{if } \ x \notin \{x_n\} \text{ and } y = x_j \end{cases}$$

is a w-distance on X.

Lemma 1. *Let X be a metric space, let p be a w-distance on X and let f be a bounded lower semicontinuous function from X into $\mathbb{R}$. Assume that c is a positive real number with $c \geq \sup f(X) - \inf f(X)$. Then a function $q : X \times X \to [0, \infty)$ defined by*

$$q(x,y) = \begin{cases} f(x) - \inf f(Mx), & \textit{if } \ y \in Mx, \\ c, & \textit{if } \ y \notin Mx \end{cases}$$

is a w-distance on X, where $Mx = \{y \in X : f(y) + p(x,y) \leq f(x)\}$.

Lemma 2. *Let X be a metric space with metric d, let p be a w-distance on X and let α be a function from X into $[0, \infty)$. Then a function $q : X \times X \to [0, \infty)$ given by*

$$q(x,y) = \max\{\alpha(x), p(x,y)\} \quad \textit{for every} \quad x, y \in X$$

is also a w-distance.

Lemma 3. *Let X be a metric space, let p be a w-distance on X, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in X and let $x, y, z \in X$. Then the following hold:*

(i) *If $p(x_n, y) \to 0$ and $p(x_n, z) \to 0$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then y=z, see* [4]*;*

(ii) *If $p(x_n, y_n) \to 0$ and $p(x_n, z) \to 0$, then $\{y_n\}$ converges to z, see* [4]*;*

(iii) *If $p(x_n, y_n) \to 0$ and $p(x_n, z_n) \to 0$, then $\{d(y_n, z_n)\}$ converges to 0.*

Lemma 4. *Let X be a metric space with metric d, let p be a w-distance on X and let $\{x_n\}$ be a sequence in X. Suppose that*

$$\lim_{n\to\infty} \sup_{m>n} \min\{p(x_n, x_m), p(x_m, x_n)\} = 0.$$

Then $\{x_n\}$ is Cauchy. In particular, the following hold:

(i) *If $\lim\limits_{n\to\infty} \sup\limits_{m>n} p(x_n, x_m) = 0$, then $\{x_n\}$ is Cauchy, see* [4]*;*

(ii) *If $\lim\limits_{n\to\infty} \sup\limits_{m>n} p(x_m, x_n) = 0$, then $\{x_n\}$ is Cauchy.*

3. Minimization Theorem

In this Section, using Theorem 2, we prove a nonconvex minimization theorem which improves Theorem 1.

Theorem 3. *Let X be a complete metric space, and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-distance p on X such that for any $u \in X$ with $f(u) > \inf_{x \in X} f(x)$, there exists $v \in X$ with $v \neq u$ and*

$$f(v) + p(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$.

Proof. Assume $f(x) > \inf f(X)$ for every $x \in X$. Put

$$Y = \{x \in X : f(x) \leq \inf f(X) + 1\}$$

and

$$Mx = \{y \in Y : f(y) + p(x, y) \leq f(x)\}$$

for every $x \in Y$ and define $q : Y \times Y \to [0, \infty)$ by

$$q(x, y) \;=\; \begin{cases} f(x) - \inf f(Mx), & \text{if} \quad y \in Mx, \\ \hfill 1, & \text{if} \quad y \notin Mx \end{cases}$$

for every $x, y \in Y$. Then, since f is lower semicontinuous, Y is closed and hence Y is complete. From Lemma 1, we have that q is a w-distance on Y. And it is clear that $y \in Mx$ and $z \in My$ imply $z \in Mx$. Let $x \in Y$ be fixed. By assumption, there exists $v \in X$ with $v \neq x$ and $f(v) + p(x, v) \leq f(x)$. Then since

$$f(v) \;\leq\; f(v) + p(x, v) \;\leq\; f(x) \;\leq\; \inf f(X) + 1,$$

we have $v \in Y$ and hence $Mx \setminus \{x\} \neq \emptyset$. So, we can choose Tx such that

$$f(Tx) \leq \frac{1}{2}\{f(x) + \inf f(Mx)\} \quad \text{and} \quad Tx \in Mx \setminus \{x\}.$$

Then, since $MTx \subseteq Mx$, we have

$$\begin{aligned} q(Tx, T^2x) &= f(Tx) - \inf f(MTx) \\ &\leq f(Tx) - \inf f(Mx) \\ &\leq \frac{1}{2}\{f(x) + \inf f(Mx)\} - \inf f(Mx) \\ &= \frac{1}{2}\{f(x) - \inf f(Mx)\} \\ &= \frac{1}{2}q(x, Tx). \end{aligned}$$

Let $\{x_n\} \subseteq Y$, $y \in Y$ with $q(x_n, y) \to 0$. By the definition of q, we may assume $y \in Mx_n$ for every $n \in \mathbb{N}$. Since $Ty \in My \subseteq Mx_n$, we have

$$q(x_n, Ty) = q(x_n, y) \to 0$$

and hence $y = Ty$ by Lemma 3. Therefore we have

$$\inf\{q(x, y) + q(x, Tx) : x \in Y\} > 0$$

for every $y \in Y$ with $y \neq Ty$. So, by Theorem 2, there exists $x_0 \in Y$ such that $x_0 = Tx_0$. This is a contradiction and this completes the proof. $\square$

Remark. Theorem 1 is not applied to the function $f(x) = x^2$. But, putting $p(x, y) = \left|\int_x^y 2|t|dt\right|$, Theorem 3 is applied to such f.

Using Theorem 3 and Example 2, we have the following corollary which generalizes the results of [5] and [10].

Corollary 1 (Takahashi [9]**).** *Let X be a complete metric space with metric d, let T be a continuous mapping from X into itself and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that for any $u \in X$ with $f(u) > \inf_{x \in X} f(x)$, there is $v \in X$ with $v \neq u$ and*

$$f(v) + \max\{d(Tu, v), d(Tu, Tv)\} \leq f(u).$$

Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$.

4. Fixed Point Theorems

In this Section, we first prove the following theorem, which is more useful than Theorem 2.

Theorem 4. *Let X be a complete metric space, let p be a w-distance on X. Let T be a mapping from X into itself and $r \in [0, 1)$ with*

$$p(Tx, T^2x) \leq rp(x, Tx)$$

for every $x \in X$. Suppose either of the following holds:

(i) $\inf\{p(x, Tx) + p(x, y) : x \in X\} > 0$ *for every $y \in X$ with $y \neq Ty$;*
(ii) *it implies $y = Ty$ that there exists a sequence $\{x_n\} \subseteq X$ such that $\{x_n\}$ and $\{Tx_n\}$ converge to y;*
(iii) *T is continuous; see* [4].

Then there exists $x_0 \in X$ such that $x_0 = Tx_0$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Proof. In the case of (i), it is already proved. Let us prove that (ii) implies (i). Let $y \in X$ with $\inf\{p(x, Tx) + p(x, y) : x \in X\} = 0$. Then there exists $\{z_n\}$ such that $p(z_n, Tz_n) \to 0$ and $p(z_n, y) \to 0$. By Lemma 3, we have $Tz_n \to y$. Since

$$\begin{aligned} p(z_n, T^2 z_n) &\leq p(z_n, Tz_n) + p(Tz_n, T^2 z_n) \\ &\leq (1 + r)p(z_n, Tz_n) \to 0, \end{aligned}$$

we have $T^2 z_n \to y$ by Lemma 3. Put $x_n = Tz_n$. Then both $\{x_n\}$ and $\{Tx_n\}$ converge to y. This implies $y = Ty$ by (ii). Hence (i) is satisfied. To complete the proof, we show that (iii) implies (ii). Let T be a continuous mapping of X. Assume that $\{x_n\}$ and $\{Tx_n\}$ converge to y. Then we have

$$Ty = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = y.$$

Therefore (ii) holds. □

In general, a w-distance p on X does not satisfy that $p(x, y) = p(y, x)$ for every $x, y \in X$. So, the condition $p(T^2x, Tx) \leq rp(Tx, x)$ for every $x \in X$, differs from the condition $p(Tx, T^2x) \leq rp(x, Tx)$. Theorem 4 is a fixed point theorem for the latter condition. We can also prove a fixed point theorem for the former condition.

Theorem 5. *Let X be a complete metric space, let p be a w-distance on X. Let T be a mapping from X into itself and $r \in [0, 1)$ such that*

$$p(T^2x, Tx) \leq rp(Tx, x)$$

for every $x \in X$. Suppose either of the following holds:

(i) *It implies $p(Ty, y) = 0$ (or equivalently $Ty = y$) that there exists a sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to y$ and $p(Tx_n, x_n) \to 0$;*
(ii) *it implies $y = Ty$ that there exists a sequence $\{x_n\} \subseteq X$ such that $\{x_n\}$ and $\{Tx_n\}$ converge to y;*
(iii) *T is continuous.*

Then there exists $x_0 \in X$ such that $x_0 = Tx_0$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Proof. First, we shall show $p(Ty, y) = 0$ is equivalent to $Ty = y$ for every $y \in X$. If $p(Ty, y) = 0$, we have

$$p(T^2y, Ty) \leq rp(Ty, y) = 0$$

and

$$p(T^2y, y) \leq p(T^2y, Ty) + p(Ty, y) = 0.$$

So, we obtain $Ty = y$ by Lemma 3. If $Ty = y$, we have

$$p(y, y) = p(T^2y, Ty) \leq rp(Ty, y) = rp(y, y)$$

and hence $p(y, y) = 0$. Next, we shall show (ii) implies (i). Let $\{x_n\}$ be a sequence in X, which converges to some point y in X and satisfies $\lim_{n\to\infty} p(Tx_n, x_n) = 0$. Then we have

$$p(T^2x_n, Tx_n) \leq rp(Tx_n, x_n) \to 0 \quad (n \to \infty)$$

and

$$\begin{aligned} p(T^2x_n, x_n) &\leq p(T^2x_n, Tx_n) + p(Tx_n, x_n) \\ &\leq rp(Tx_n, x_n) + p(Tx_n, x_n) \\ &= (1 + r)p(Tx_n, x_n) \to 0 \quad (n \to \infty). \end{aligned}$$

By Lemma 3 and $\{x_n\}$ converges to y, we have $\{Tx_n\}$ also converges to y. So, from (ii), y is a fixed point of T and hence (i) holds. It is from the proof of Theorem 4 that (iii) implies (ii). So, to complete the proof, we prove T has a fixed point in the case of (i). Let $u \in X$ and define

$$u_n = T^n u \quad \text{for any} \quad n \in \mathbb{N}.$$

Then we have, for any $n \in \mathbb{N}$,

$$p(u_{n+1}, u_n) \leq rp(u_n, u_{n-1}) \leq \cdots \leq r^n p(u_1, u).$$

So, if $m > n$,

$$\begin{aligned} p(u_m, u_n) &\leq p(u_m, u_{m-1}) + \cdots + p(u_{n+1}, u_n) \\ &\leq r^{m-1}p(u_1, u) + \cdots + r^n p(u_1, u) \\ &\leq \frac{r^n}{1-r} p(u_1, u). \end{aligned}$$

By Lemma 4, $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $x_0 \in X$. And we have

$$p(Tu_n, u_n) \leq r^n p(u_1, u) \to 0.$$

So, by assumption, we have $p(Tx_0, x_0) = 0$. Therefore x_0 is a fixed point of T. This completes the proof. □

Now, we prove Ćirić's fixed point theorem by two methods.

Corollary 2 (Ćirić [2]). *Let X be a complete metric space with metric d, and let T be a mapping from X into itself. Suppose T is quasi-contraction, i.e., there exists $r \in [0, 1)$ such that*

$$d(Tx, Ty) \leq r \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for every $x, y \in X$. Then T has a unique fixed point.

Proof by Theorem 4. By lemma 2 in [2], $\{x, Tx, T^2x, \cdots\}$ is bounded for every $x \in X$. Hence we can define a function $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \max\{\text{diam}\{x, Tx, T^2x, \cdots\}, d(x, y)\}$$

for every $x, y \in X$. By Lemma 2, p is a w-distance on X. Let $x \in X$. Then we have, using lemma 1 in [2],

$$\begin{aligned} p(Tx, T^2x) &= \text{diam}\{Tx, T^2x, T^3x, \cdots\} \\ &= \sup_{n \in \mathbf{N}} \text{diam}\{Tx, T^2x, T^3x, \cdots T^nx\} \\ &\leq \sup_{n \in \mathbf{N}} r \cdot \text{diam}\{x, Tx, T^2x, \cdots T^nx\} \\ &= r \cdot \text{diam}\{x, Tx, T^2x, \cdots\} \\ &= r \cdot p(x, Tx). \end{aligned}$$

Assume $\{x_n\}$ and $\{Tx_n\}$ converge to y. Since T is quasi-contraction,

$$d(Tx_n, Ty) \leq r \max\{d(x_n, y), d(x_n, Tx_n), d(y, Ty), d(x_n, Ty), d(y, Tx_n)\}$$

for any $n \in \mathbb{N}$. So,

$$\begin{aligned} d(y, Ty) &\leq r \max\{d(y, y), d(y, y), d(y, Ty), d(y, Ty), d(y, y)\} \\ &= rd(y, Ty) \end{aligned}$$

and hence $y = Ty$. By Theorem 4, there exists a fixed point z of T. Clearly, a fixed point is unique. This completes the proof. □

Proof by Theorem 5. We can define a function $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \sup\{d(T^kx, y) : k \in \mathbb{N} \cup \{0\}\}$$

for every $x, y \in X$. By Example 3, p is a w-distance on X. Let $x \in X$. Then we have, using lemma 1 in [2],

$$\begin{aligned} p(T^2x, Tx) &= \sup\{d(T^kx, Tx) : k = 2, 3, 4, \cdots\} \\ &\leq r \cdot \sup\{d(T^kx, x) : k = 1, 2, 3, \cdots\} \\ &= r \cdot p(x, Tx). \end{aligned}$$

So, by Theorem 5, there exists a fixed point z of T. This completes the proof. □

5. Metric Completeness

In this Section, we discuss a characterization of metric completeness. First, we give a definition. A mapping $T : X \to X$ is called weakly contractive if there exist a w-distance p on X and $r \in [0, 1)$ such that $p(Tx, Ty) \le rp(x, y)$ for every $x, y \in X$. The following Theorem was proved in [7]. We give another proof of "if" part and two proofs of "only if" part.

Theorem 6 ([7]). *Let X be a metric space. Then X is complete if and only if every weakly contractive mapping from X into itself has a fixed point in X.*

Proof of "if" part. Assume that X is not complete. Then there exists a sequence $\{x_n\}$ in X satisfying the following conditions:

(i) $\{x_n\}$ is Cauchy;
(ii) $\{x_n\}$ does not converge;
(iii) $x_i \neq x_j$ if $i \neq j$.

A function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) = \begin{cases} 2^{-i} + 2^{-j}, & \text{if } \ x = x_i \text{ and } y = x_j, \\ 2^{-i} + 1, & \text{if } \ x = x_i \text{ and } y \notin \{x_n\}, \\ 1 + 2^{-j}, & \text{if } \ x \notin \{x_n\} \text{ and } y = x_j \end{cases}$$

is a w-distance on X, by Example 4. Define a mapping T from X into itself as follows:

$$Tx = \begin{cases} x_{i+1}, & \text{if } \ x = x_i, \\ x_1, & \text{otherwise.} \end{cases}$$

Then we have $p(Tx, Ty) \le \frac{1}{2} p(x, y)$ for every $x, y \in X$. But, T has not a fixed point in X. This completes the proof. $\square$

Proof of "only if" part by Theorem 4. Clearly,

$$p(Tx, T^2x) \le rp(x, Tx)$$

for every $x \in X$. Let $y \in X$ with $y \neq Ty$ be fixed. Assume that there exists $\{x_n\}$ such that

$$\lim_{n\to\infty} \{p(x_n, y) + p(x_n, Tx_n)\} = 0.$$

Then we have

$$\begin{aligned} p(x_n, Ty) &\le p(x_n, Tx_n) + p(Tx_n, Ty) \\ &\le p(x_n, Tx_n) + rp(x_n, y) \ \to \ 0. \end{aligned}$$

Then, by Lemma 3, we have $Ty = y$. This is a contradiction. Hence, we have

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0.$$

By Theorem 4, T has a fixed point. $\square$

Proof of "only if" part by Theorem 5. Clearly,

$$p(T^2x, Tx) \le rp(Tx, x)$$

for every $x \in X$. Let $\{x_n\}$ be a sequence in X which converges to some point y in X and satisfies $\lim_{n\to\infty} p(Tx_n, x_n) = 0$. Let $k \in \mathbb{N}$ be fixed. Then we have

$$\begin{aligned} p(T^k y, x_n) &\le p(T^k y, T^k x_n) + \sum_{i=1}^{k-1} p(T^{i+1}x_n, T^i x_n) + p(Tx_n, x_n) \\ &\le r^k p(y, x_n) + \sum_{i=0}^{k-1} r^i p(Tx_n, x_n) \\ &= r^k p(y, x_n) + \frac{1-r^k}{1-r} p(Tx_n, x_n) \end{aligned}$$

and hence $p(T^k y, y) \le r^k p(y, y)$. So, we obtain

$$p(T^k y, Ty) \le rp(T^{k-1}y, y) \le r^k p(y, y).$$

By Lemma 3, we have $Ty = y$. Therefore, by Theorem 5, T has a fixed point. $\square$

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